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Steady finite-amplitude waves on a horizontal seabed of arbitrary depth

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From shallow-water gravity wave theories it is shown that the velocity field in the whole fluid domain can be reconstructed using an analytic transformation (a renormalization). The resulting velocity field satisfies the Laplace equation exactly, which is not the case for shallow-water approximations. Applying the renormalization to the first-order shallow-water solution of limited accuracy, gives accurate simple solutions for both long and short waves, even for large amplitudes. The KdV and Airy solutions are special limiting cases.

1. Introduction

Surface wave problems have long been of interest to mathematicians, physicists and engineers. Mathematical theories of gravity waves provide an efficient qualitative understanding of many phenomena, and they are also able to give good quantitative predictions. The underlying mathematical difficulty stems from the nonlinearity of the equations at the free surface. Generally, to obtain approximate analytical expressions, perturbation techniques are employed. There are two main theories based on these techniques.

One technique, Poincaré's small-parameter method, consists of looking for a solution near the rest position of the system in the form of a power series in a small parameter. For gravity waves, this expansion is called the Stokes theory (Stoker 1957). The first-order approximation leads to the usual linear theory (Airy's theory), the solutions of which can be expressed (in Cartesian coordinates) in terms of circular and hyperbolic functions. At higher orders, inhomogeneous linear equations must be solved, and it is found that each order adds another harmonic. This yields a solution in the form of a Fourier series. The results are such that, when the wavelength is allowed to tend to infinity, in water of finite depth, the amplitude must vanish if solutions are required to remain finite. For this reason these solutions are described as short linear waves, and the theory is also known as that of short waves. The Stokes theory, in particular, is incapable of describing solitary waves.

The inability of Poincaré's small-parameter method to deal with long waves of finite amplitude, and in particular solitary waves, means that a second theory must be employed. This is referred to as shallow-water or long-wave theory. To allow for large scales, a distortion is introduced, characterized by a small parameter, in the horizontal and time variables. A solution close to the rest position is then sought that can be expanded in a power series in the small parameter. For progressive waves,

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this yields the so-called cnoidal solutions (because they can be written in terms of Jacobian elliptic cn-functions) which admit solitary waves as a limiting case. In the other limiting case where the elliptic functions are reduced to circular functions, the amplitude of the wave vanishes. The long-wave theory therefore cannot describe the solutions of the Stokes theory.

Thus, the two theories have different domains of validity (Littman 1957). Mathematically, Poincaré's small-parameter method generates a regular perturbation problem: the resulting series are convergent (Levi-Cività 1925). In contrast, the technique used in the shallow-water theory leads to a singular perturbation problem: the resulting series are divergent (Germain 1967). It is not possible to pass from the solutions of one theory (at a given order of approximation) to those of the other.

All theoretical investigations of gravity wave problems use, more or less, these two fundamental techniques.

The purpose of this investigation is to show how it is possible to unify these two types of theories. In § 2 we give an exact general analytical formula (a renormalization formula) which reconstructs the velocity field, from the velocity potential at the bottom only, within the entire fluid domain. The solution obtained satisfies exactly the Laplace equation and bottom impermeability equation in a fluid of arbitrary depth. To illustrate the method, in § 3 we apply the formula to the Korteweg–de Vries (KdV) solution for a permanent wave. Moreover, we use the boundary conditions at the free surface to improve the surface profile and the relations between the wave parameters. We show that the renormalization extends the validity of the shallow-water approximation to deep water. In § 4, we compare the results with 'exact' numerical solutions to the same problem. It is shown that the renormalization also increases the range of validity of the solution accuracy from small to large amplitudes.

2. Renormalization principle

2.1. *Two-dimensional surface gravity wave equations*

In this paper, we consider the particular case of two-dimensional irrotational wave motion on the surface of an homogeneous incompressible inviscid fluid with a constant depth h. The velocity potential φ satisfies to the Laplace equation for $0 \leqslant y \leqslant h + \eta(x, t)$

$$\varphi_{xx} + \varphi_{yy} = 0, \tag{2.1}$$

with boundary condition at the bottom $y = 0$

$$\varphi_y = 0, \tag{2.2}$$

where x, y, t denote the horizontal, upward vertical and time variables respectively, and $\eta(x, t)$ is the surface elevation from the mean level $y = h$. The boundary conditions at the free surface $y = h + \eta(x, t)$, for a stationary wave, are

$$g\eta - C\varphi_x + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_y^2 = \beta, \tag{2.3}$$

$$\varphi_y = -C\eta_x + \eta_x\varphi_x, \tag{2.4}$$

where g is the acceleration due to gravity, C is the phase velocity and β is a Bernoulli constant.

2.2. *General solution of the Laplace equation*

The problem consists of solving the Laplace equation with nonlinear boundary conditions. The Laplace equation and the bottom impermeability condition can be solved exactly. Indeed, the most general solution of the Laplace equation which respects the bottom impermeability is

$$\varphi(x, y, t) = \tfrac{1}{2}\hat{\varphi}(x + \mathrm{i}y, t) + \tfrac{1}{2}\hat{\varphi}(x - \mathrm{i}y, t), \tag{2.5}$$

where $\hat{\varphi}(x, t) \equiv \varphi(x, 0, t)$ is the potential at the bottom and $\mathrm{i}^2 = -1$. Relation (2.5) is a special form of d'Alembert's solution for the wave equation adapted to the Laplace equation. From (2.5) the stream function ψ and the velocity components $u = \varphi_x = \psi_y$, $v = \varphi_y = -\psi_x$ are

$$\psi(x, y, t) = \frac{1}{2\mathrm{i}}\hat{\varphi}(x + \mathrm{i}y, t) - \frac{1}{2\mathrm{i}}\hat{\varphi}(x - \mathrm{i}y, t), \tag{2.6}$$

$$u(x, y, t) = \tfrac{1}{2}\hat{u}(x + \mathrm{i}y, t) + \tfrac{1}{2}\hat{u}(x - \mathrm{i}y, t), \tag{2.7}$$

$$v(x, y, t) = \frac{\mathrm{i}}{2}\hat{u}(x + \mathrm{i}y, t) - \frac{\mathrm{i}}{2}\hat{u}(x - \mathrm{i}y, t), \tag{2.8}$$

where $\hat{u}(x, t) \equiv \hat{\varphi}_x(x, t)$ is the horizontal velocity at the bottom. Note that with (2.6) we choose to take $\psi \equiv 0$ at the bottom. The solution obtained for φ satisfies *ipso facto* the Laplace equation and the bottom impermeability condition. The function $\hat{\varphi}$ provides a description of a wave field both in the fluid interior and at the bottom, so it is sufficient to know the potential at the bottom to reconstruct it in the whole domain. Hence, with any function $\hat{\varphi}$ and with the formula (2.5), we obtain a function φ which satisfies the Laplace equation and the bottom impermeability identically, and so a velocity field of zero divergence is obtained.

After solving the Laplace equation and the bottom impermeability, it is also necessary to satisfy the boundary conditions at the free surface. To obtain approximate analytical solutions, the perturbation methods mentioned above can be used. These methods give approximations of φ and *a fortiori* of $\hat{\varphi}$. The use of the reconstruction formula (2.5) with the Stokes solutions of short waves has no practical interest. The reason is that these solutions already satisfy exactly the Laplace equation and (2.5) reconstructs the same solutions. On the other hand, the shallow-water-type approximations do not solve the Laplace equation exactly, and (2.5) provides an efficient means to improve the approximations. Indeed, the Taylor expansion of $\hat{\varphi}(x + \mathrm{i}y, t)$ around $y = 0$ is

$$\hat{\varphi}(x + \mathrm{i}y, t) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\left[\frac{\partial^n \hat{\varphi}(x + \mathrm{i}y, t)}{\partial y^n}\right]_{y=0} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!}\frac{\partial^n \hat{\varphi}(x, t)}{\partial x^n}, \tag{2.9}$$

and from this expansion (2.5) becomes

$$\varphi(x, y, t) = \sum_{n=0}^{\infty}(-1)^n \frac{y^{2n}}{(2n)!}\frac{\partial^{2n}\hat{\varphi}(x, t)}{\partial x^{2n}}. \tag{2.10}$$

Equation (2.10) is the development of φ due to Korteweg & de Vries (1895). Every long-wave approximation (e.g. Boussinesq 1872; Serre 1953) uses a truncated version of (2.10) and gives approximations of $\hat{\varphi}$. From these approximations, it is then easy to obtain φ with (2.5). Since (2.5) transforms a Taylor expansion into its closed original form, it is called a renormalization formula.

2.3. *A generalization: variable bottoms*

The renormalization principle can be extended to more complicated two-dimensional domains. Let us consider a non-horizontal bottom given by the equation $y = \zeta(x)$.

Incompressibility, irrotationality and bottom impermeability lead to the following equations:

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \text{for} \quad \zeta \leqslant y \leqslant h + \eta, \tag{2.11}$$

$$\varphi_y - \zeta_x \varphi_x = 0 \quad \text{at} \quad y = \zeta. \tag{2.12}$$

To solve these equations, we can consider a conformal mapping $(x, y) \rightarrow (X, Y)$ of the physical domain $\zeta \leqslant y \leqslant h + \eta$ (or $\zeta \leqslant y \leqslant h$ for small-amplitude waves) into the band $0 \leqslant Y \leqslant h$. Laplace's equation and bottom impermeability become

$$\Phi_{XX} + \Phi_{YY} = 0 \quad \text{for} \quad 0 \leqslant Y \leqslant h, \tag{2.13}$$

$$\Phi_Y = 0 \quad \text{at} \quad Y = 0, \tag{2.14}$$

where $\Phi(X, Y) \equiv \varphi[x(X, Y), y(X, Y)]$ and to determine the conformal mapping we need to solve

$$y_{XX} + y_{YY} = 0 \quad \text{for} \quad 0 \leqslant Y \leqslant h, \tag{2.15}$$

$$y = \zeta \quad \text{at} \quad Y = 0, \tag{2.16}$$

$$y = h + \eta \quad \text{at} \quad Y = h, \tag{2.17}$$

with the Cauchy–Riemann relations $x_X = y_Y$, $x_Y = -y_X$. Except for simple geometries, the conformal mapping cannot be known exactly and explicitly. However, singular perturbation methods can be used to approximately solve the conformal mapping equations (2.15)–(2.17), and renormalization can be used in the following way.

The most general solution of the Laplace equation (2.13) which satisfies the bottom boundary condition (2.14) is

$$\Phi(X, Y) = \tfrac{1}{2}\hat{\Phi}(X + \mathrm{i}Y) + \tfrac{1}{2}\hat{\Phi}(X - \mathrm{i}Y), \tag{2.18}$$

where $\hat{\Phi} = \Phi(X, 0)$. Similarly, the most general solution of the Laplace equation (2.15) – together with the Cauchy–Riemann relations – which satisfies the bottom boundary condition (2.16) is

$$x(X, Y) = \tfrac{1}{2}\hat{x}(X + \mathrm{i}Y) + \tfrac{1}{2}\hat{x}(X - \mathrm{i}Y) + \frac{\mathrm{i}}{2}\zeta[\hat{x}(X + \mathrm{i}Y)] - \frac{\mathrm{i}}{2}\zeta[\hat{x}(X - \mathrm{i}Y)], \tag{2.19}$$

$$y(X, Y) = \frac{1}{2\mathrm{i}}\hat{x}(X + \mathrm{i}Y) - \frac{1}{2\mathrm{i}}\hat{x}(X - \mathrm{i}Y) + \tfrac{1}{2}\zeta[\hat{x}(X + \mathrm{i}Y)] + \tfrac{1}{2}\zeta[\hat{x}(X - \mathrm{i}Y)], \tag{2.20}$$

where $\hat{x} = x(X, 0)$. Hence, since approximations of $\hat{\Phi}$ and $\hat{x}$ are known, relations (2.18)–(2.20) can be used to build a new approximation which satisfies exactly the Laplace equation (2.11) and the bottom impermeability (2.12).

Solving conformal mapping equations with a singular perturbation method consists, physically, in considering a 'slowly' varing bottom. This assumption is always made in shallow-water theories, and for short waves it is used, for example, to derive mild-slope equations (Mei 1989).

3. Solutions for shallow and deep water

In this section, we present the application of the renormalization formula to the stationary solutions of the first-order shallow-water theory. There exist many equivalent variants of this theory (Mei 1989). One of them yields the Korteweg & de Vries (1895) (KdV) equation

$$\hat{u}_t + C_0\,\hat{u}_x + \tfrac{3}{2}\,\hat{u}\,\hat{u}_x + \tfrac{1}{6}C_0 h^2\,\hat{u}_{xxx} = 0, \tag{3.1}$$

with $C_0^2 = gh$; η is related to the horizontal speed at the bottom by $\hat{u} = \hat{\varphi}_x \simeq C_0\eta/h$. This is a consequence of the independence of φ from y (at this order of approximation). Stationary solutions depend on the single variable $\theta = x - Ct + \delta$, where C is the phase velocity and δ is a constant phase shift.

3.1. *Renormalization of linear very long waves*

The stationary solution of the linearized KdV equation is

$$\hat{\varphi} = \mathscr{A} \sin(k\theta), \qquad \eta = a\cos(k\theta), \tag{3.2}$$

with

$$k = 2\pi/L, \quad C/C_0 = 1 - (kh)^2/6, \quad \mathscr{A} = aC_0/kh, \tag{3.3}$$

where L is the wavelength and a is the wave amplitude. This sinusoidal approximation is linear and weakly dispersive, in contrast to Airy's short-wave solution which is strongly dispersive. Note also that (3.2) does not satisfy the Laplace equation exactly. Hence, if we apply the renormalization formula (2.5) to this approximation, we will obtain an improved solution which satisfies the Laplace equation identically. Indeed, the renormalization of (3.2) yields

$$\begin{aligned}\varphi(\theta, y) &= \tfrac{1}{2}\hat{\varphi}(\theta + \mathrm{i}y) + \tfrac{1}{2}\hat{\varphi}(\theta - \mathrm{i}y)\\ &= \tfrac{1}{2}\mathscr{A}\sin[k(\theta + \mathrm{i}y)] + \tfrac{1}{2}\mathscr{A}\sin[k(\theta - \mathrm{i}y)]\\ &= \mathscr{A}\sin(k\theta)\cosh(ky).\end{aligned} \tag{3.4}$$

With the help of the renormalization, we have rebuilt the Airy solution of short waves. It is obvious that the solution obtained satisfies the Laplace equation exactly, and that is a better approximation than (3.2). Of course, the renormalization of (3.2) is of minor practical interest, but it is demonstrated here, as a simple example, that renormalized approximations necessarily have velocity fields whose divergence is zero.

3.2. *Renormalization of cnoidal waves*

We shall now apply the renormalization to the exact stationary solution of the KdV equation. Because this solution involves nonlinear terms, we shall obtain a better approximation than with the linear approximation (3.2).

The stationary solution to the KdV equation can be expressed using Jacobi's elliptic functions sn, cn, dn, Z (Z is the zeta-function) and the elliptic integrals of the first and second kind K, E of parameter m (Abramowitz & Stegun 1965). The cnoidal solution is generally written with the cn-function, but to determine the velocity potential, it is more convenient – and strictly equivalent – to use the dn-function. The stationary solution of the KdV equation is hence

$$\hat{\varphi} = \frac{A}{\kappa} Z(\kappa\theta|m), \quad \hat{u} = A\left[\mathrm{dn}^2(\kappa\theta|m) - E/K\right], \quad \eta = a\frac{\mathrm{dn}^2(\kappa\theta|m) - E/K}{1 - E/K}, \tag{3.5}$$

where κ is a kind of wavenumber, a is the wave amplitude and A is a parameter related to the maximum speed. These parameters are linked by relations

$$\frac{a}{h}\frac{C_0}{A} = 1 - \frac{E}{K}, \quad \kappa^2 h^2 = \frac{3}{4}\frac{A}{C_0}, \quad \frac{C}{C_0} = 1 + \frac{a}{2h}\frac{2 - m - 3E/K}{1 - E/K}, \quad \kappa L = 2K, \tag{3.6}$$

where L is the wavelength. Note that m can be viewed as an Ursell parameter (Dean & Dalrymple 1991). Note also that it is more useful, in shallow-water theories, to use the total wave elevation H (i.e the crest to trough elevation) which is related to the wave amplitude by $H = ma/(1 - E/K)$ (the choice of a instead of H is discussed in § 3.4.3). Solution (3.5) is weakly nonlinear and weakly dispersive, and includes (3.2) as the limiting case for $m \ll 1$.

Applying reconstruction formula (2.5) to the exact KdV solution (3.5) gives (see Abramowitz & Stegun 1965, #17.4.35–36)

$$\begin{aligned}\varphi &= \frac{A}{2\kappa} Z[\kappa(\theta + \mathrm{i}y)|m] + \frac{A}{2\kappa} Z[\kappa(\theta - \mathrm{i}y)|m] \\ &= \frac{A}{\kappa}\left[Z(\kappa\theta|m) + \frac{m\, \mathrm{sn}(\kappa\theta|m)\,\mathrm{cn}(\kappa\theta|m)\,\mathrm{dn}(\kappa\theta|m)\,\mathrm{sn}^2(\kappa y|m_1)}{\mathrm{cn}^2(\kappa y|m_1) + m\,\mathrm{sn}^2(\kappa\theta|m)\,\mathrm{sn}^2(\kappa y|m_1)}\right], \end{aligned} \tag{3.7}$$

where $m_1 = 1 - m$. Equation (3.7) is the renormalized Korteweg & de Vries solution (RKdV). From (3.7) the stream function and the velocity components are

$$\psi = \frac{A}{\kappa}\left\{ \frac{\mathrm{d}^2\, \mathrm{s}_1\, \mathrm{c}_1\, \mathrm{d}_1}{\mathrm{c}_1^2 + m\,\mathrm{s}^2\,\mathrm{s}_1^2} - \mathrm{z}_1 - \frac{\pi\kappa y}{2KK'} \right\}, \tag{3.8}$$

$$u = A\left\{ \frac{\mathrm{d}^2\,\mathrm{c}_1^2\,\mathrm{d}_1^2 - m^2\,\mathrm{s}^2\,\mathrm{c}^2\,\mathrm{s}_1^2}{[\mathrm{c}_1^2 + m\,\mathrm{s}^2\,\mathrm{s}_1^2]^2} - \frac{E}{K} \right\}, \tag{3.9}$$

$$v = 2mA\left\{ \frac{\mathrm{s}\,\mathrm{c}\,\mathrm{d}\,\mathrm{s}_1\,\mathrm{c}_1\,\mathrm{d}_1}{[\mathrm{c}_1^2 + m\,\mathrm{s}^2\,\mathrm{s}_1^2]^2} \right\}, \tag{3.10}$$

where $K' = K(m_1)$ and the notation

$$\begin{bmatrix} \mathrm{s} \\ \mathrm{c} \\ \mathrm{d} \\ \mathrm{z} \end{bmatrix} = \begin{bmatrix} \mathrm{sn} \\ \mathrm{cn} \\ \mathrm{dn} \\ Z \end{bmatrix} (\kappa\theta|m), \qquad \begin{bmatrix} \mathrm{s}_1 \\ \mathrm{c}_1 \\ \mathrm{d}_1 \\ \mathrm{z}_1 \end{bmatrix} = \begin{bmatrix} \mathrm{sn} \\ \mathrm{cn} \\ \mathrm{dn} \\ Z \end{bmatrix} (\kappa y|m_1), \tag{3.11}$$

has been used. Since relations (3.7)–(3.10) are derived from (3.5) via (2.5), all of them satisfy the Laplace equation exactly and the bottom impermeability condition. We have then obtained a velocity field with a divergence equal to zero. This point can be easily verified by considering the Fourier expansion of (3.7): the RKdV potential is periodic for $m \neq 1$ and is thus expandable in the Fourier series

$$\varphi = \frac{\pi A}{\kappa K}\sum_{n=1}^{\infty} \mathrm{cosech}\left(\frac{n\pi K'}{K}\right) \sin\left(\frac{n\pi\kappa\theta}{K}\right) \cosh\left(\frac{n\pi\kappa y}{K}\right). \tag{3.12}$$

It is obvious that (3.12) satisfies the Laplace equation exactly and the bottom impermeability condition. Note that (3.12) is obtained from the Fourier series of the potential at the bottom (Abramowitz & Stegun 1965, #17.4.38), and, in the second step, by application of the renormalization. This series is practically impossible to derive directly from (3.7). It is an illustration of the power of the renormalization. Note also that (3.12) has no practical value in computing the solution for long waves. It is more efficient to compute it directly from (3.7). In the limiting case of solitary waves ($m = 1$), the potential is

$$\varphi = \frac{A}{\kappa}\frac{\tanh(\kappa\theta)}{1 - \mathrm{sech}^2(\kappa\theta)\sin^2(\kappa y)}, \tag{3.13}$$

and one can verify that is an exact solution of the Laplace equation. Since (3.12) is a Stokes-type expansion of short waves in deep water, (3.7) is a unified approximation of φ for shallow and deep water. However, this unification is partial because relations between parameters (3.6) are only valid for small-amplitude long waves. The renormalized solution (3.7) improves the solution accuracy for long waves (Clamond 1998), but to allow a correct description of short waves, relations between parameters have to be redefined.

3.3. *New evaluation of the surface elevation*

In the first-order shallow-water theory, η is evaluated with a velocity field independent of y. After renormalization, φ is not constant along the vertical and its value at the surface is significantly changed. The consequence is that η must be changed too, which implies that equations at the free surface have to be solved. It is always possible, for steady or unsteady flows, to obtain a new evaluation of η with the Bernoulli equation. However, for stationary waves, it is more convenient to derive η from the flow conservation law

$$\eta_t + \tilde{\psi}_x = 0, \tag{3.14}$$

where $\tilde{\psi} \equiv \psi(x, h+\eta, t)$ is the stream function at the surface. This equation can be integrated, for a progressive wave, in the form

$$\eta = C^{-1}\tilde{\psi} - \alpha, \tag{3.15}$$

where α is an integration constant. The definition of η (3.15) is implicit and its effective determination requires numerical computations. In the shallow-water theory at first-order, weakly nonlinear effects are taken into account by quadratic terms. It is thus consistent to derive an explicit approximation of η from (3.15) by using an expansion of $\tilde{\psi}$ truncated at the quadratic term,

$$\eta \simeq C^{-1}\left[\,\psi(\theta, h) + \eta\, u(\theta, h)\,\right] - \alpha. \tag{3.16}$$

Hence

$$\eta \simeq \frac{\psi(\theta, h) - \alpha\, C}{C - u(\theta, h)}. \tag{3.17}$$

For simplicity, we now only consider the quadratic approximation of η (3.17). The re-evaluation of the surface elevation involves some new relations between parameters. After renormalization the wavelength is still $L = 2K/\kappa$, but to complete the solution, it is necessary to find five other relations between parameters.

3.4. *Redefinition of the mean level and of the amplitude*

After the renormalization, φ and η are changed and redefinitions of the wave amplitude and of the mean level are necessary.

3.4.1. *Frame and mean surface elevation*

We choose to express the solution in the frame without mean velocity at the bottom and we define η as the surface elevation from the mean depth h. Hence, we impose the conditions

$$\langle \hat{u} \rangle = 0, \qquad \langle \eta \rangle = 0, \tag{3.18}$$

where the Eulerian average operator $\langle\ \rangle$ is defined by

$$\langle \bullet \rangle = \frac{1}{L}\int_{-L/2}^{+L/2} \bullet \,\mathrm{d}\theta. \tag{3.19}$$

With the definition (3.9) of u, the condition $\langle \hat{u} \rangle = 0$ is satisfied identically. The condition $\langle \eta \rangle = 0$ in (3.17) gives an equation for α,

$$\alpha = \frac{1}{C} \left\langle \frac{\psi(\theta, h)}{C - u(\theta, h)} \right\rangle \Big/ \left\langle \frac{1}{C - u(\theta, h)} \right\rangle . \tag{3.20}$$

The condition $\langle \eta \rangle = 0$, applied to the Bernoulli equation, gives an equation for β,

$$\beta = \tfrac{1}{2} \left\langle \tilde{u}^2 + \tilde{v}^2 \right\rangle - C \left\langle \tilde{u} \right\rangle , \tag{3.21}$$

where $(\tilde{u}, \tilde{v})$ are velocity components at the surface.

3.4.2. *Relation between the wave amplitude and the maximum speed*

We define the wave amplitude a as the maximum of the surface elevation from the mean level h, hence

$$a = \frac{\psi(0, h) - \alpha C}{C - u(0, h)}, \tag{3.22}$$

with

$$\psi(0, h) = \frac{A}{\kappa} \left[\frac{\operatorname{sn}(\kappa h | m_1) \operatorname{dn}(\kappa h | m_1)}{\operatorname{cn}(\kappa h | m_1)} - Z(\kappa h | m_1) - \frac{\pi \kappa h}{2 K K'} \right],$$

$$u(0, h) = A \left[\frac{\operatorname{dn}^2(\kappa h | m_1)}{\operatorname{cn}^2(\kappa h | m_1)} - \frac{E}{K} \right].$$

3.4.3. *Trough height and total wave height*

We can also define the trough height b as the minimum of the surface elevation from the mean level (i.e. $b = -\eta(L/2)$). As was the case previously, the definition of η gives the definition of b

$$b = \frac{\psi(L/2, h) - \alpha C}{u(L/2, h) - C}, \tag{3.23}$$

with

$$\psi(L/2, h) = \frac{A}{\kappa} \left[\frac{m_1 \operatorname{sn}(\kappa h | m_1) \operatorname{cn}(\kappa h | m_1)}{\operatorname{dn}(\kappa h | m_1)} - Z(\kappa h | m_1) - \frac{\pi \kappa h}{2 K K'} \right],$$

$$u(L/2, h) = A \left[\frac{m_1 \operatorname{cn}^2(\kappa h | m_1)}{\operatorname{dn}^2(\kappa h | m_1)} - \frac{E}{K} \right].$$

In classical shallow-water theories, it is useful to introduce the total wave height $H = a + b$. In these theories, there are some algebraic relations between parameters, and the introduction of H has some practical interest (Fenton 1979). In Stokes's theories of short waves, there are also algebraic relations with amplitudes and use of H is also convenient (Fenton 1990). The RKdV solution is fully nonlinear and relations between parameters involve transcendental functions. Thus, use of the parameter H is less convenient than in classical theories. Moreover, the definition of a implies that the important notion of mean level has already been defined, i.e. that it is not required for the definition of H. Hence a includes more information about the wave than H. For these reasons, use of a instead of H appears 'more natural' and 'more physical'. However, these considerations are semantic and of secondary importance.

3.5. *Enclosure relations*

To find the two missing relations between parameters, the free-surface isobarity condition has to be used. We have a solution which satisfies exactly the Laplace equation, the bottom boundary condition and the kinematic condition at the free surface (with the exact implicit definition of η). Unfortunately, this solution does not satisfy Bernoulli's equation exactly.

One way to find these relations could be a least-square minimization between the RKdV approximation and the exact solution. Another way is to use an averaged Lagrangian minimization (Witham 1974). However, due to the complexity of the Jacobian function, these approaches lead to intractable algebra.

A simpler, and also rigorous, alternative is to use an interpolation between the exact solution and the RKdV approximation. Since there are two free parameters, we shall require that RKdV is equal to the exact solution at two special points. Points are chosen to obtain as simple expressions as possible, and where relations involve physical parameters directly. The best candidates are the crest and the trough of the wave.

Interpolation is a simple way to derive enclosure relations. For non-progressive waves, interpolation is less consistent. For unsteady flows, the most consistent method is probably use of the average Lagrangian.

3.5.1. *Limiting amplitude relation*

At the crest of the wave (i.e. at $\theta = 0$), the RKdV approximation is taken equal to the exact solution, and Bernoulli's equation gives

$$C - \left(C^2 + 2\beta - 2ga\right)^{1/2} = \tilde{u}(0) = A\left\{\frac{\mathrm{dn}^2\left[\kappa(h+a)|m_1\right]}{\mathrm{cn}^2\left[\kappa(h+a)|m_1\right]} - \frac{E}{K}\right\}. \tag{3.24}$$

This relation imposes a limitation on the wave amplitude (i.e. $C^2 + 2\beta \geqslant 2ga$).

3.5.2. *Nonlinear dispersion relation*

At the minimum trough (i.e. at $\theta = L/2$), the RKdV approximation is also taken equal to the exact solution, and Bernoulli's equation gives

$$C - \left(C^2 + 2\beta + 2gb\right)^{1/2} = \tilde{u}(L/2) = A\left\{\frac{m_1\,\mathrm{cn}^2\left[\kappa(h-b)|m_1\right]}{\mathrm{dn}^2\left[\kappa(h-b)|m_1\right]} - \frac{E}{K}\right\}. \tag{3.25}$$

Equation (3.25) is the last relation sought. We shall see in §4 that (3.25) can be considered as a nonlinear dispersion relation.

4. Comparisons with exact solutions

We now compare the renormalized solution with exact numerical solutions. There exist various efficient algorithms to compute elliptic functions (Abramowitz & Stegun 1965). Fenton & Gardiner-Garden (1982) gave an efficient formula for the case of m close to unity. These algorithms provide easy and fast methods to compute RKdV solutions.

4.1. *Solitary waves*

The renormalized solution (3.7)–(3.10) for a solitary wave is significantly simplified:

$$\varphi = \frac{A}{\kappa}\frac{\tanh(\kappa\theta)}{1 - \mathrm{sech}^2(\kappa\theta)\sin^2(\kappa y)}, \tag{4.1}$$

$$\psi = \frac{A}{2\kappa}\frac{\mathrm{sech}^2(\kappa\theta)\sin(2\kappa y)}{1 - \mathrm{sech}^2(\kappa\theta)\sin^2(\kappa y)}, \tag{4.2}$$

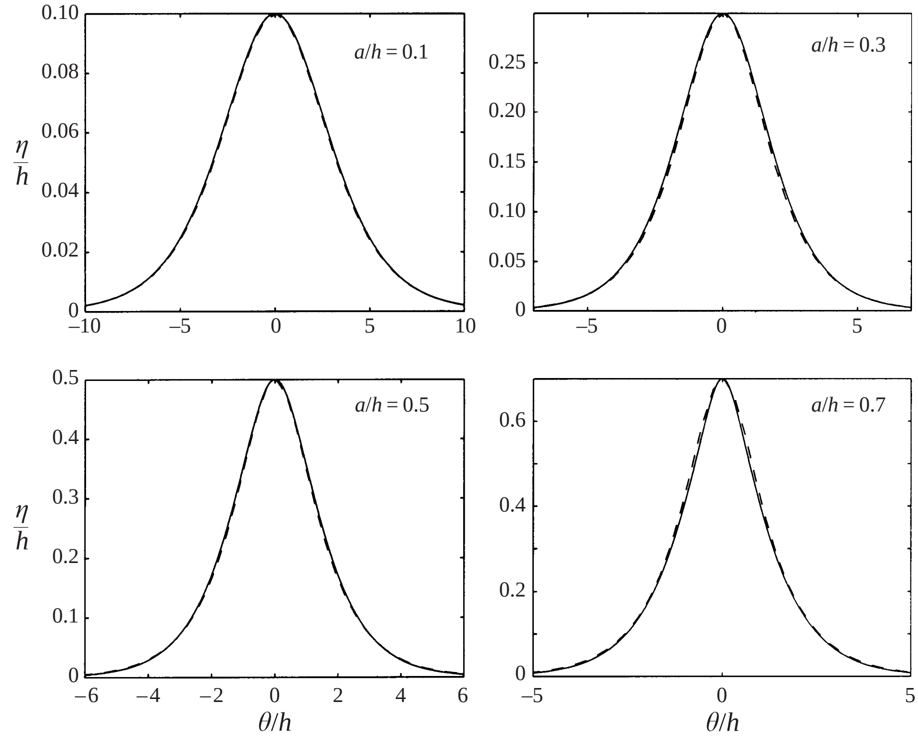

FIGURE 1. Surface elevations of a solitary wave. —, Exact; – –, RKdV.

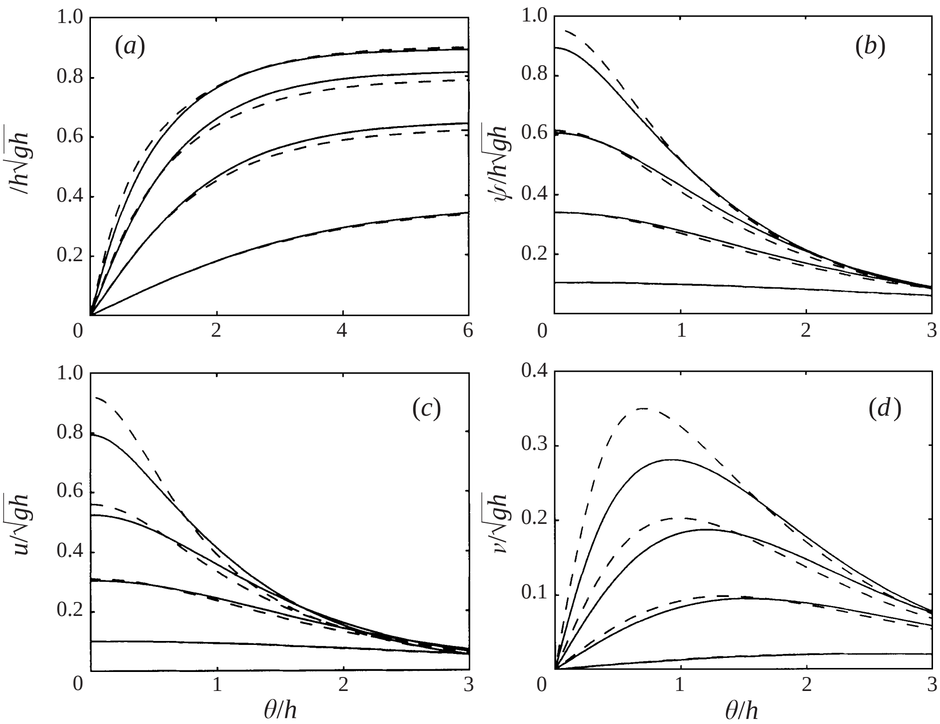

FIGURE 2. Kinematic components at the surface of a solitary wave. —, Exact; – –, RKdV, for $a/h = 0.1, 0.3, 0.5, 0.7$. (*a*) Speed potential, (*b*) stream function, (*c*) horizontal speed, (*d*) vertical speed.

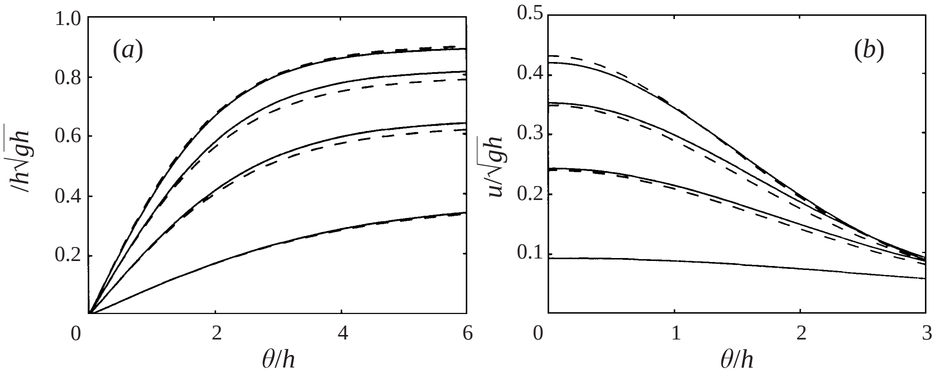

FIGURE 3. Kinematic components at the bottom for a solitary wave. —, Exact; – –, RKdV, for $a/h = 0.1, 0.3, 0.5, 0.7$. (*a*) Speed potential, (*b*) horizontal speed.

$$u = A\frac{\operatorname{sech}^2(\kappa\theta)\cos(2\kappa y) + \operatorname{sech}^4(\kappa\theta)\sin^2(\kappa y)}{\left[1 - \operatorname{sech}^2(\kappa\theta)\sin^2(\kappa y)\right]^2}, \tag{4.3}$$

$$v = A\frac{\tanh(\kappa\theta)\operatorname{sech}^2(\kappa\theta)\sin(2\kappa y)}{\left[1 - \operatorname{sech}^2(\kappa\theta)\sin^2(\kappa y)\right]^2}. \tag{4.4}$$

Relations between parameters are

$$m = 1, \qquad L = \infty, \qquad \alpha = 0, \qquad \beta = 0, \qquad b = 0,$$

$$a = \frac{A}{\kappa}\frac{\tan(\kappa h)}{C - A\sec^2(\kappa h)}, \qquad C - \left(C^2 - 2ga\right)^{1/2} = A\sec^2\left[\kappa(h + a)\right],$$

and the dispersion relation (3.25) is satisfied identically. To obtain the relation in this limiting case, we use for $m \to 1$ (i.e. for $m_1 \ll 1$) the first-order Taylor expansion of (3.23) and $\tilde{u}(L/2)$

$$\frac{b\kappa C}{A} = -\frac{m_1}{4}\sin(2\kappa h) + O(m_1^2), \qquad \tilde{u}(L/2) = \frac{m_1 A}{2}\cos(2\kappa h) + O(m_1^2),$$

and (3.25) for $m_1 = 0$ gives after some cancellation

$$\frac{C^2}{gh} = \frac{\tan(2\kappa h)}{2\kappa h}. \tag{4.5}$$

The relation (4.5) was first obtained by Stokes and it is exact (Lamb 1932). The KdV relation between C and κ is just an approximation of (4.5) limited at κ^2, and the second-order shallow-water solution is an approximation limited at κ^4 (Laitone 1960). Therefore, we can conclude that the renormalization has improved the solution to take into account strong dispersion effects.

Tanaka (1986) gave an algorithm to compute numerically the exact solitary wave solution. We then compare this exact solution to the RKdV approximation.

Exact and RKdV solutions for surface elevations are very close, even for large amplitudes (figure 1). The velocity potential, the stream function and the horizontal component of velocity at the surface are also well approximated by the RKdV solution. The vertical component of velocity is also accurate, but the relative error is larger than for the horizontal component of velocity for large amplitudes (figure 2). However, the RKdV approximation is considerably better than the simple KdV approximation. At the bottom, RKdV is very accurate (figure 3). For large amplitudes,

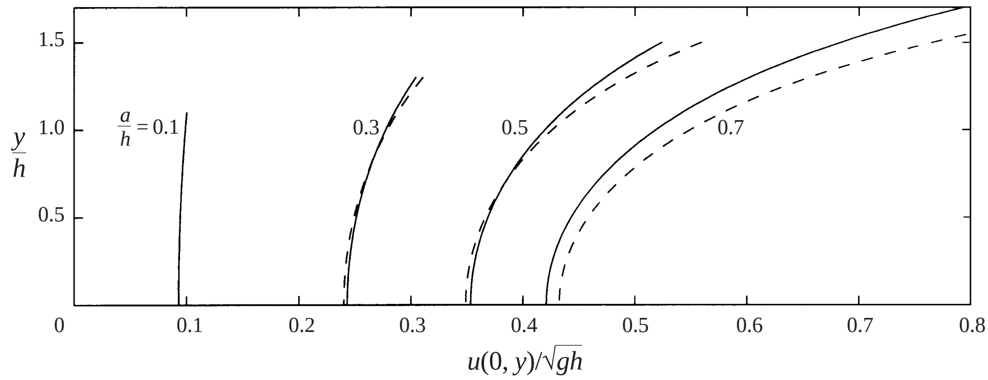

FIGURE 4. Fluid velocity under the crest of a solitary wave. —, Exact; – –, RKdV.

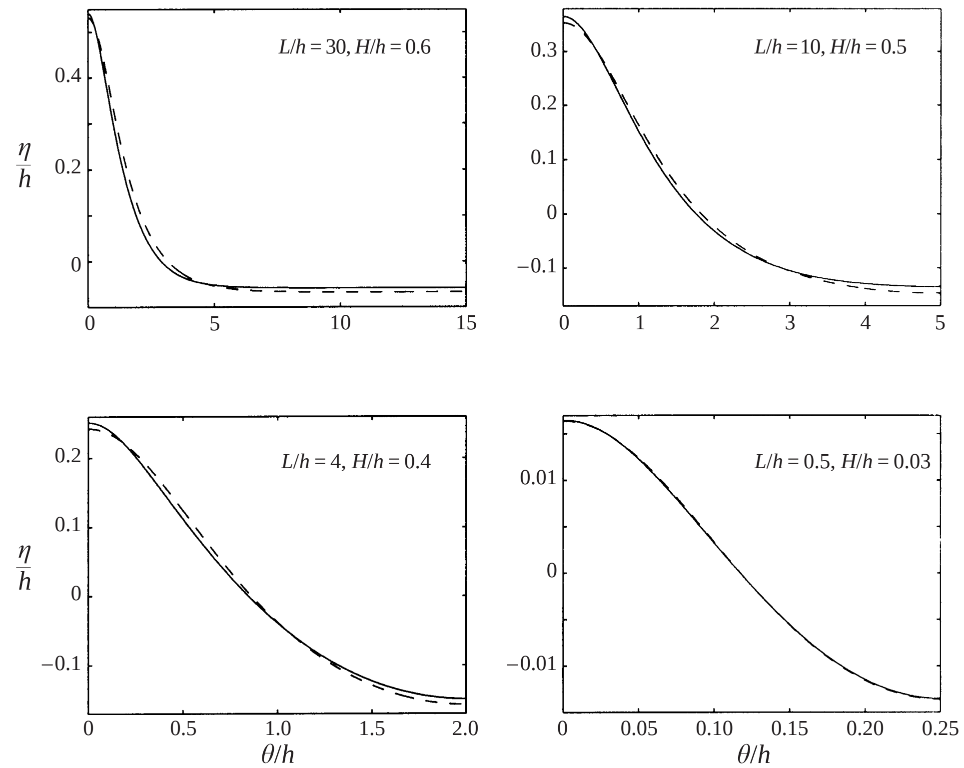

FIGURE 5. Surface elevations. —, Exact; – –, RKdV.

the velocity field of a solitary wave has significant vertical variations, and RKdV is very close to the exact solution (figure 4). This is an important consequence of the zero divergence due to the renormalization.

RKdV accuracy is comparable with high-order approximations, such as the ninth-order solution of Fenton (1972), but RKdV has the advantage of simplicity. RKdV is accurate for large amplitudes, but it is not so efficient for limiting solitons. This case could be investigated by the renormalization of Fenton's solution.

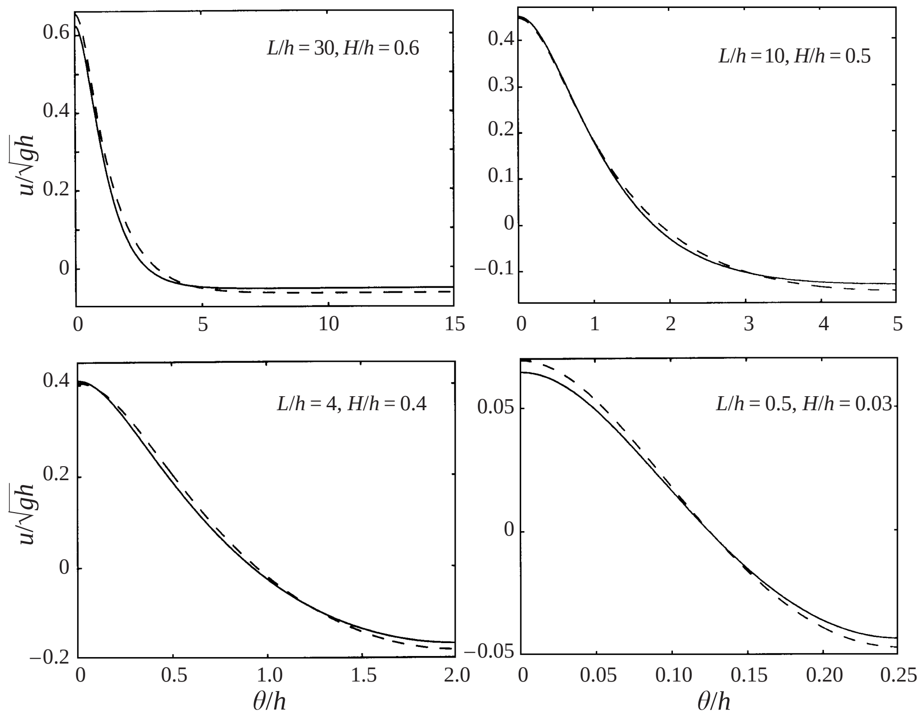

FIGURE 6. Horizontal velocity at the surface. —, Exact; – –, RKdV.

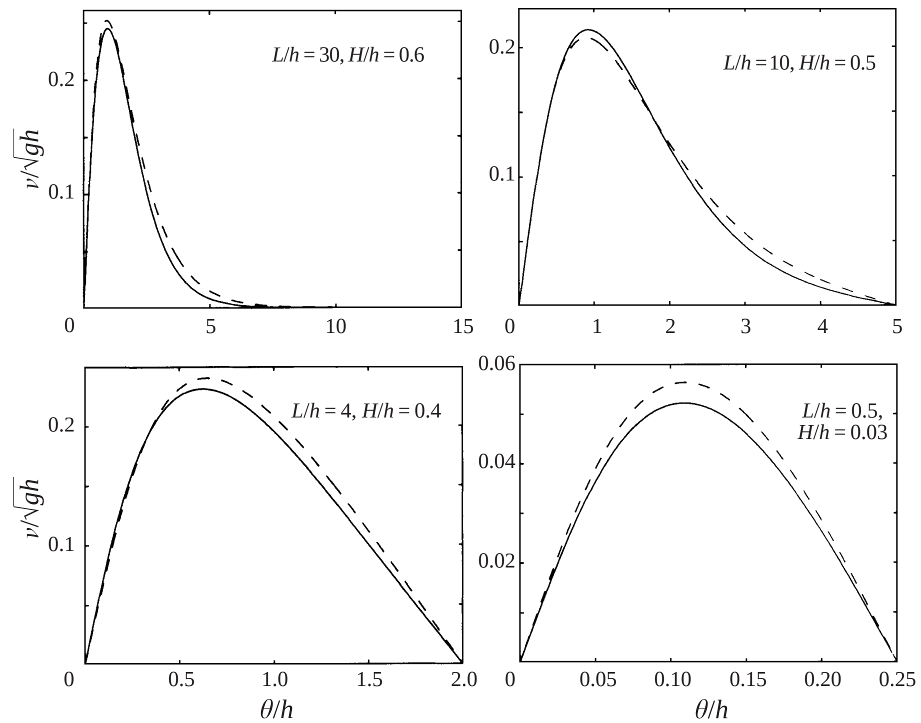

FIGURE 7. Vertical velocity at the surface. —, Exact; – –, RKdV.

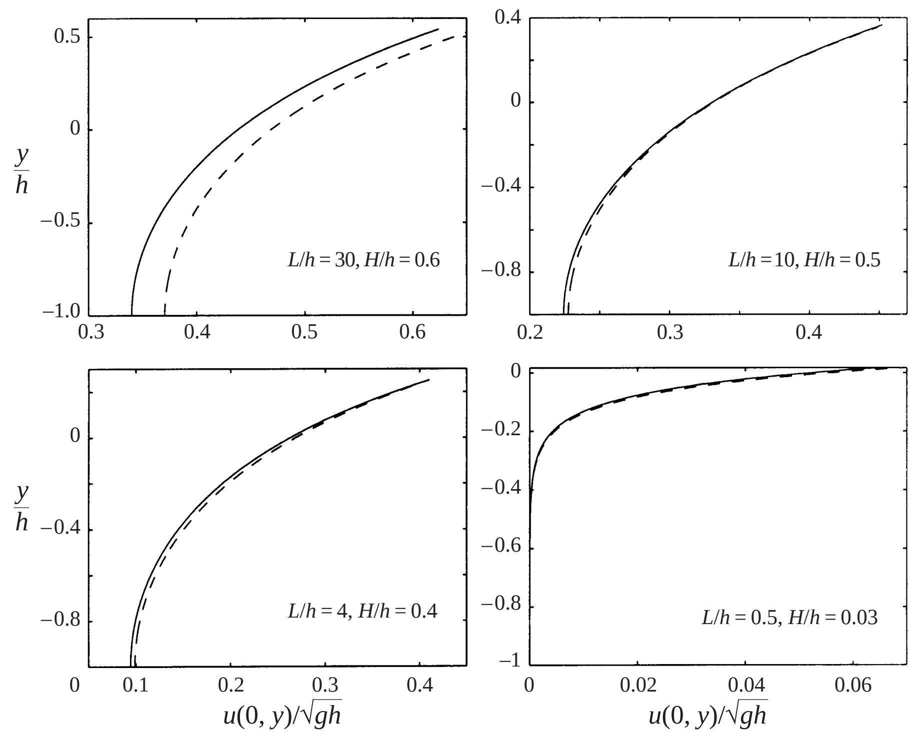

FIGURE 8. Fluid velocity under the crest. —, Exact; – –, RKdV.

4.2. *Periodic waves*

We compare RKdV periodic approximations to exact numerical solutions obtained with the method of Fenton (1988).

RKdV surface elevations are very accurate for long, intermediate, short and very short waves, even for relatively large amplitudes (figure 5). The horizontal speed at the surface is also well described by RKdV in every case (figure 6). For the vertical speed at the surface, a good agreement is found and, remarkably, the accuracy increases when the wavelength decreases (figure 7). The fluid velocity is well described by RKdV, specially for short waves (figure 8). Of course, RKdV approximations are better for small-amplitude waves but they are still good for large amplitudes, as can be seen on figures 5–8 where only large amplitudes are plotted.

For long waves, RKdV accuracy is comparable with high-order theories such as Fenton's fifth-order solution (Fenton 1979). To obtain his expansion, Fenton assumed that the parameter $\epsilon \sim H/(mh)$ is small. This solution is accurate for long waves with finite amplitudes, but diverges disastrously for short waves (i.e. when $m \ll 1$). On the other hand, RKdV does not fail for short waves, *a contrario* it is a very good approximation. This fact can be understood by considering, for small-amplitude short waves (i.e. for $m \ll 1$), the Taylor expansion of RKdV around $m = 0$ (Abramowitz & Stegun 1965, #16.13.1–3). This expansion, limited at the first term, gives

$$\varphi \simeq \frac{a\,C}{\sinh(kh)} \sin(k\theta)\cosh(ky), \qquad \eta \simeq a\cos(k\theta), \tag{4.6}$$

$$k = \frac{2\pi}{L} \simeq 2\kappa, \quad \alpha \simeq 0, \quad \beta \simeq 0, \quad b \simeq a, \quad \frac{C^2}{gh} \simeq \frac{\tanh(kh)}{kh}. \tag{4.7}$$

This approximation is exactly the Airy solution for small-amplitude short waves. It is now clear that the renormalized approximations are unified solutions for short and long waves. Airy's dispersion relation and Stokes's relation (4.5) are derived from (3.25), and for this reason, (3.25) can be considered as a general nonlinear dispersion relation. Except for these two limiting cases, (3.25) does not reduce to simple relations.

For short waves, RKdV's accuracy is also comparable to that of high-order Stokes such expansions as the fifth-order solutions of Skjelbreia & Hendrickson (1961) or Fenton (1985). Since RKdV is simpler than these expansions and can deal with long waves as well, it is an attractive alternative to classical high-order theories.

5. Conclusion

With a simple analytical formula we have obtained, from first-order shallow-water theory, an improved analytical approximation of velocity potentials and of surface elevations of progressive waves. This 'renormalized' approximation is a unified solution for shallow and deep water: it includes the Airy and Korteweg & de Vries theories as limiting cases. The renormalization also extends the solution validity from small to large amplitudes.

The renormalized approximation is very close to the exact solution. However, since this approximation satisfies the Bernoulli equation exactly at only two points, this very good agreement is probably partly accidental.

The renormalization can be applied to non-progressive waves, such as analytic solutions of the nonlinear Schrödinger equation. This type of renormalization could also be efficiently used for several analogous two-dimensional problems, such as multi-layer fluids and variable bottoms.

This appears to be a promising powerful technique, and further investigations are necessary to judge the potential of the method. In particular, if a similar technique can be derived for three-dimensional flows, it could provide a powerful improvement on existing theories derived with singular perturbation methods.

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